

Kaluza-Klein vacuum multi-black holes in five-dimensions

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Abstract

We investigate five-dimensional vacuum solutions which represent rotating multi-black holes in asymptotically Kaluza-Klein spacetimes. We show that multi-black holes rotate maximally along extra dimension, and stationary configurations in vacuum are achieved by the balance of the gravitational attraction force and repulsive force caused by the rotations of black holes. We also show that each black hole can have the different topology of the lens space in addition to the spherical topology, and mass of black holes are quantized by the size of extra dimension and horizon topology.

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I. INTRODUCTION

Kaluza-Klein black holes, which have compact extra dimensions, are interesting spacetimes in a class of higher-dimensional black holes. In particular, there exist a variety of exact solutions of five-dimensional squashed Kaluza-Klein black holes, which have squashed S^3 horizons [1–19]. The solutions behave as fully five-dimensional black holes in the vicinity of the horizon, while they behave as four-dimensional black holes in the region far away from the horizon. The squashed Kaluza-Klein black holes asymptote to four-dimensional flat spacetimes with a twisted S^1 as a compactified extra dimension, and we can regard a series of these solutions as models of realistic higher-dimensional black holes.

In the Einstein-Maxwell theory in four dimensions, it is well known that extremal charged black holes can make multi-configurations [20, 21]. In five-dimensions, multi-black hole solutions are obtained both in asymptotically flat spacetimes [22, 23] and in asymptotically Kaluza-Klein spacetimes [22, 24–26]. These solutions are achieved by the balance of gravitational attraction and electrical repulsion. Until now, any regular asymptotically flat vacuum multi-black hole solutions with spherical horizons in four or five dimensions have not been found [27, 28].¹

In contrast to the asymptotically flat case, in asymptotically Kaluza-Klein spacetimes, in this paper, we show that the metrics found by Clement [32] describe five-dimensional regular maximally rotating vacuum multi-black holes with a twisted S^1 as an extra dimension. Each black hole has a horizon with the topology of lens space $L(n; 1) = S^3/\mathbb{Z}_n$. If the size of extra dimension L is fixed, the regularity condition requires the quantization of black hole mass by nL .

This paper is organized as follows. In Sec.II, we present explicit forms of solutions and constraints between parameters of the solutions. Section III is devoted to an investigation of conserved charges, asymptotic structures of the solutions, and the regularity at the horizons. We conclude our studies with discussions in Sec.IV.

¹ For non-spherical horizons in five dimensions, black Saturns and black di-rings are found as vacuum exact solutions [29–31].

II. SOLUTIONS

We consider rotating multi-black hole solutions satisfying the five-dimensional vacuum Einstein equation, $R_{\mu\nu} = 0$. The metric is written as

$$ds^2 = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2) + 2 \left[(H^{-1} - 1) dt + \frac{L}{2\sqrt{2}} d\psi + \boldsymbol{\omega} \right]^2, \quad (1)$$

where

$$H = 1 + \sum_i \frac{m_i}{|\mathbf{R} - \mathbf{R}_i|} \quad (2)$$

is the harmonic function on the three-dimensional Euclid space with point sources located at $\mathbf{R} = \mathbf{R}_i := (x_i, y_i, z_i)$. The 1-form $\boldsymbol{\omega}$, which is determined by

$$\nabla \times \boldsymbol{\omega} = \nabla H, \quad (3)$$

has the explicit form

$$\boldsymbol{\omega} = \sum_i m_i \frac{z - z_i}{|\mathbf{R} - \mathbf{R}_i|} \frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2}. \quad (4)$$

In the expressions (1)-(4), m_i and L are constants. The solutions (1) with (2) and (4) can be obtained by uplifting the four-dimensional equally charged dyonic Majumdar-Papapetrou solutions with a constant dilaton field to the five-dimensional spacetimes [32]. (See Appendix A for detail discussion.)² As will be shown later, $\mathbf{R} = \mathbf{R}_i$ are black hole horizons.

From the requirements for the absence of naked singularity on and outside the black hole horizons, the parameters are restricted to the range

$$m_i > 0. \quad (5)$$

We will see later that the regularity of horizons requires the parameters m_i to be quantized by the size of the compactified dimension L at infinity in the form

$$m_i = \frac{n_i L}{2\sqrt{2}}, \quad (6)$$

where n_i are the natural numbers.

² In the single black hole case, $m_1 = m$ and $m_i = 0$ ($i \geq 2$), the solution (1) coincides with an extremally rotating vacuum squashed Kaluza-Klein black hole solution with a degenerate horizon [2]. The solution (1) was also obtained in the context of the ten-dimensional $N = 1$ supergravity [33]. However, to the best of our knowledge, properties of the solution (1) like asymptotic structures and a smoothness of horizons have not been discussed.

III. PROPERTIES

A. Basic properties

It is clear that the metric (1) admits two Killing vector fields,

$$\xi_{(t)} = \partial/\partial t \quad \text{and} \quad \xi_{(\psi)} = \partial/\partial \psi. \quad (7)$$

Timelike Killing vectors which are timelike at infinity are not unique because the Killing vector along the compact extra dimension $\xi_{(\psi)}$ has a finite norm at the infinity. This fact is quite different from the asymptotically flat case, where only $\xi_{(t)}$ is timelike at the infinity. Fortunately, however, we can select out the timelike Killing vector $\xi_{(t)}$ that is hypersurface orthogonal at the infinity among them. We define the Komar mass M associated with $\xi_{(t)}$, and obtain as

$$M = \frac{-3}{32\pi} \int_{\infty} dS_{\mu\nu} \nabla^{\mu} \xi_{(t)}^{\nu} = \frac{3L \sum_i m_i}{4\pi} \mathcal{A}_{S^3}, \quad (8)$$

where \mathcal{A}_{S^3} denotes the area of a unit S^3 . We also obtain the angular momentum J_{ψ} associated with the spacelike Killing vector $\xi_{(\psi)}$ as

$$J^{\psi} = \frac{1}{16\pi} \int_{\infty} dS_{\mu\nu} \nabla^{\mu} \xi_{(\psi)}^{\nu} = \frac{L^2 \sum_i m_i}{4\sqrt{2}\pi} \mathcal{A}_{S^3}. \quad (9)$$

We see that the spacetime (1) is rotating along the extra dimension. Because the present solutions are vacuum solutions, the mass and angular momentum can be assigned to each black hole

$$M_i = \frac{3Lm_i}{4\pi} \mathcal{A}_{S^3}, \quad (10)$$

$$J_i^{\psi} = \frac{L^2 m_i}{4\sqrt{2}\pi} \mathcal{A}_{S^3}, \quad (11)$$

by taking integral for the closed surface surrounding each black hole.

Substituting (6) into (10) and (11), we obtain a relation between the mass and the angular momentum as

$$(J_i^{\psi})^2 = \frac{2\sqrt{2}}{27\pi n_i} M_i^3. \quad (12)$$

This relation means that each black hole is maximally rotating. We can also obtain the relation

$$\frac{J_i^{\psi}}{M_i} = \frac{L}{3\sqrt{2}}. \quad (13)$$

Total mass and total angular momentum are given by the summations,

$$M = \sum_i M_i, \quad J^\psi = \sum_i J_i^\psi, \quad (14)$$

which satisfy the condition

$$(J^\psi)^2 = \frac{2\sqrt{2}}{27\pi n} M^3, \quad (15)$$

where $n = \sum_i n_i$.

With respect to the timelike Killing vector $\xi_{(t)}$, we define the ergosurfaces where the Killing vector becomes null, i.e.,

$$g_{tt} = (H^{-1} - 2)^2 - 2 = 0. \quad (16)$$

In the single black hole case, $m_1 = m$ and $m_i = 0$ ($i \geq 2$), the equation (16) reduces to

$$g_{tt} = \frac{2m^2 - R^2}{(R + m)^2} = 0. \quad (17)$$

Then, the ergosurface exists at $R = \sqrt{2}m$. In general, since $g_{tt}(\mathbf{R} = \mathbf{R}_i) = 2 > 0$ and $g_{tt}(\infty) = -1 < 0$ for the range of parameters (5), there always exist ergoregions around the black hole horizons. It depends on the configuration of point sources whether the ergoregions are connected or not [26].

B. Asymptotic structure

We assume that point sources exist in a bounded domain. In the far region from the domain, the harmonic function H and the 1-form ω behave as

$$H \simeq 1 + \frac{\sum_i m_i}{R} + O(R^{-2}), \quad (18)$$

$$\omega \simeq \left(\sum_i m_i \right) \cos \theta d\phi + O(R^{-1}). \quad (19)$$

Then, using (6), we see that the metric behaves as

$$ds^2 \simeq - \left(1 + \frac{m}{R}\right)^{-2} dt^2 + \left(1 + \frac{m}{R}\right)^2 (dR^2 + R^2 d\Omega_{\mathbb{S}^2}^2) + \frac{n^2 L^2}{4} \left(-\frac{dt}{R} + \frac{d\psi}{n} + \cos \theta d\phi \right)^2, \quad (20)$$

where $d\Omega_{\mathbb{S}^2}^2 = d\theta^2 + \sin^2\theta d\phi^2$ denotes the metric of the unit two-sphere, and $m = \sum_i m_i$. The metric behaves as a single extremely rotating Kaluza-Klein black hole [1, 2]. In the limit $R \rightarrow \infty$, we see that the metric approaches as

$$ds^2 \rightarrow -dt^2 + dR^2 + R^2 d\Omega_{\mathbb{S}^2}^2 + \frac{n^2 L^2}{4} \left(\frac{d\psi}{n} + \cos\theta d\phi \right)^2. \quad (21)$$

The asymptotic structure of the spacetime (1) is asymptotically locally flat, i.e., the metric asymptotes to a twisted constant S^1 fiber bundle over the four-dimensional Minkowski spacetime, and the spatial infinity has the structure of an S^1 bundle over an S^2 such that it is the lens space $L(n; 1) = S^3/\mathbb{Z}_n$ [25, 26]. We see that the size of a twisted S^1 fiber as an extra dimension takes the constant value L everywhere.

C. Near horizon

For simplicity, we restrict ourselves to the cases of two-black holes, i.e., $m_i = 0$ ($i \geq 3$). Without loss of generality, we can put the locations of two point sources as $\mathbf{R}_1 = (0, 0, 0)$ and $\mathbf{R}_2 = (0, 0, a)$, where the constant a denotes the separation between two black holes.

In this case, the metric is

$$ds^2 = -H^{-2}dt^2 + H^2 (dR^2 + R^2 d\Omega_{\mathbb{S}^2}^2) + 2 \left[(H^{-1} - 1) dt + \frac{L}{2\sqrt{2}} d\psi + \boldsymbol{\omega} \right]^2, \quad (22)$$

where H and $\boldsymbol{\omega}$ are given by

$$H = 1 + \frac{m_1}{R} + \frac{m_2}{\sqrt{R^2 + a^2 - 2aR \cos\theta}}, \quad (23)$$

$$\boldsymbol{\omega} = \left(m_1 \cos\theta + m_2 \frac{R \cos\theta - a}{\sqrt{R^2 + a^2 - 2aR \cos\theta}} \right) d\phi, \quad (24)$$

respectively. The coordinates run the ranges of $-\infty < t < \infty$, $0 < R < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \psi \leq 4\pi$.

In the coordinate system $(t, R, \theta, \phi, \psi)$, the metric (22) diverges at the locations of two point sources, i.e., $\mathbf{R} = \mathbf{R}_1$ ($R = 0$) and $\mathbf{R} = \mathbf{R}_2$ ($R = a, \theta = 0$).

In order to remove apparent divergences at $R = 0$, we introduce new coordinates (v, ψ') such that

$$dv = dt + H^2 dR + W d\theta, \quad (25)$$

$$d\psi' = d\psi - \frac{2\sqrt{2}}{L} (dt + H dR + V d\theta), \quad (26)$$

where the functions W and V are given by

$$W(R, \theta) = \int dR \frac{\partial}{\partial \theta} (H^2), \quad (27)$$

$$V(R, \theta) = \int dR \frac{\partial}{\partial \theta} H, \quad (28)$$

respectively. Then, the metric (22) takes the form of

$$ds^2 = -H^{-2} (dv - W d\theta)^2 + 2dR (dv - W d\theta) + H^2 R^2 d\Omega_{\mathbb{S}^2}^2 + 2 \left[H^{-1} dv + \boldsymbol{\omega} + (V - H^{-1}W) d\theta + \frac{L}{2\sqrt{2}} d\psi' \right]^2. \quad (29)$$

In the neighborhood of $R = 0$, the functions H , W , V , and the 1-form $\boldsymbol{\omega}$ can be expanded by power series in R , and leading orders are

$$H \simeq \frac{m_1}{R} + \mathcal{O}(1), \quad (30)$$

$$W \simeq -\frac{2m_1 m_2 \sin \theta}{a^2} R + \mathcal{O}(R^2), \quad (31)$$

$$V \simeq -\frac{m_2 \sin \theta}{2a^2} R^2 + \mathcal{O}(R^3), \quad (32)$$

$$\boldsymbol{\omega} \simeq (m_1 \cos \theta - m_2 + \mathcal{O}(R^2)) d\phi, \quad (33)$$

where we have chosen the integral constants of W and V given by (27) and (28) suitably.

Then, near $R = 0$, the metric (29) behaves as

$$ds^2 \simeq \frac{R^2}{m_1^2} dv^2 + 2dv dR + m_1^2 \left[d\Omega_{\mathbb{S}^2}^2 + 2 \left(\frac{L}{2\sqrt{2}m_1} d\psi'' + \cos \theta d\phi \right)^2 \right] + 4R \left[\frac{m_1 m_2 \sin \theta}{a^2} dR d\theta + \left(dv + \frac{2m_1 m_2 \sin \theta}{a^2} R d\theta \right) \left(\frac{L}{2\sqrt{2}m_1} d\psi'' + \cos \theta d\phi \right) \right] + \mathcal{O}(R^3), \quad (34)$$

where we have used

$$d\psi'' = d\psi' - \frac{2\sqrt{2}}{L} m_2 d\phi. \quad (35)$$

If the factor $2\sqrt{2}m_1/L$ is a natural number, say n_1 , the induced metric on the three-dimensional spatial cross section of $R = 0$ with a time slice is

$$ds^2|_{R=0} = \frac{n_1^2 L^2}{8} \left[d\Omega_{\mathbb{S}^2}^2 + 2 \left(\frac{d\psi''}{n_1} + \cos \theta d\phi \right)^2 \right]. \quad (36)$$

That is, if the mass quantization condition (6) holds, the $R = 0$ surface admits the smooth metric of the squashed lens space $L(n_1; 1) = S^3/\mathbb{Z}_{n_1}$. The area of the surface is

$$\mathcal{A}|_{R=0} = \frac{n_1^2 L^3}{2} \mathcal{A}_{S^3}. \quad (37)$$

Under the condition (6), we see that $R = 0$ is a null surface where the metric (29) is regular and each component is an analytic function of R . Therefore the metric (29) gives analytic extension across the surface $R = 0$. By the same discussion, we see that the metric (22) also admits analytic extension across the surface $\mathbf{R} = \mathbf{R}_2$.

We also see that $\eta = \partial_v$ is a Killing vector field that becomes null at $R = 0$. Furthermore, η is hypersurface orthogonal to the surface $R = 0$, i.e., $\eta_\mu dx^\mu = g_{vR} dR = dR$ there. These mean that the null hypersurface $R = 0$ is a Killing horizon. Similarly, $\mathbf{R} = \mathbf{R}_2$ is also a Killing horizon. Hence, we can see that the solutions (22) with (23) and (24) describe Kaluza-Klein multi-black holes, which have smooth Killing horizons without singularity on and outside the black hole horizons. The topology of each black hole horizon is the lens spaces $L(n_i; 1)$. Since the ϕ - ψ part of the metric is positive definite, it is clear that no closed timelike curve exists. Near each horizon limit, the metric (22) approaches the $L(n_i; 1)$ bundle over the AdS_2 space at the horizon [34, 35].

IV. SUMMARY AND DISCUSSIONS

We have investigated extremely rotating Kaluza-Klein multi-black hole solutions in the five-dimensional pure Einstein theory given by the metric (1) with (2) and (4). The metric asymptotes to the effectively four-dimensional spacetime and the size of the compactified extra dimension takes the constant value everywhere. We have shown that each black hole has a smooth horizon and its topology is the lens space. Furthermore, the mass and the angular momentum of the black hole satisfy the extremality condition and the horizon size of each black hole is quantized by the size of the compactified dimension. To sum up, the exact solutions describe five-dimensional regular vacuum rotating Kaluza-Klein multi-black holes.

In the solutions, for each black hole, the ratio of the mass and the angular momentum is determined rigidly by a value of order of unity. We can interpret that this comes from the force balance between gravitational force and spin-spin repulsive force between black holes.

This corresponds to the balance of gravitational force and Coulomb repulsive force after Kaluza-Klein reduction. Furthermore, the black hole mass should be quantized by the size of extra dimension L from the regularity of the horizon. The minimum size of black hole, which is comparable to L , exists. Then, we cannot get asymptotically flat solutions from the present solutions by taking a limit $L \rightarrow \infty$ keeping the black hole mass constant. This is consistent with the fact that no vacuum multi-black hole has been found in five-dimensional asymptotically flat spacetime. The asymptotic structure of the solution with a compact dimension affects the existence of the multi black holes. Whether more general solutions exist or not is open question.

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Appendix A: Kaluza-Klein reduction

We consider the Kaluza-Klein parametrization of a five-dimensional metric,

$$ds^2 = e^{-\Phi/3} ds_{(4)}^2 + e^{2\Phi/3} \left(\frac{L}{2} d\psi + \mathbf{A} \right)^2 \quad (\text{A1})$$

which admits the Killing vector field ∂_ψ , where $ds_{(4)}^2$, $\mathbf{F} = d\mathbf{A}$, and Φ are identified with the four-dimensional metric, the four-dimensional Maxwell field, and the dilaton field, respectively. The metric $ds_{(4)}^2$ and two fields \mathbf{F} , Φ satisfy the field equations of the four-dimensional Einstein-Maxwell-dilaton theory with the action

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g_{(4)}} \left(R_{(4)} - \frac{1}{6} \partial_\mu \Phi \partial^\mu \Phi - \frac{e^\Phi}{4} F_{\mu\nu} F^{\mu\nu} \right). \quad (\text{A2})$$

After the Kaluza-Klein reduction of the metric (1) with respect to the Killing vector field ∂_ψ , we find

$$ds_{(4)}^2 = -H^{-2} dt^2 + H^2 (dx^2 + dy^2 + dz^2), \quad (\text{A3})$$

$$\mathbf{F} = \sqrt{2} d(H^{-1} dt + \boldsymbol{\omega}), \quad (\text{A4})$$

$$\Phi = \text{const.}, \quad (\text{A5})$$

where the function H is given by (2) and the four-dimensional Maxwell field \mathbf{F} satisfies the relation $F_{\mu\nu}F^{\mu\nu} = 0$. We see that (A3)-(A5) coincide with the four-dimensional equal charged dyonic Majumdar-Papapetrou solution with a constant dilaton field.

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